Worcester County Mathematics League

Varsity Meet 1
October 4, 2017

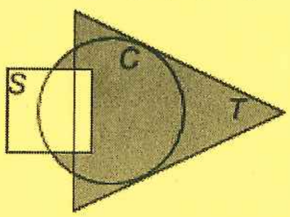
Coaches' Copy
Rounds, Answers, and Solutions

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 4, 2017 ANSWER KEY

Round 1:
1. 20 (Southbridge)
2. -4 (Douglas)
3. 760,537 (AMSA)

Round 2:
1. 51.5 or $51\frac{1}{2}$ or $\frac{103}{2}$ (Bartlett)
2. 320 (St. John's)
3. 18 (Mass Academy)

Round 3:
1. P or {2, 4, 6, 8, 10} (Douglas)
2. 

(Bromfield)
3. 64 (Leicester)

Round 4:
1. 9 (Holy Name)
2. 81π (Burncoat)
3. $2\sqrt{3}-3$ (Westboro)

Round 5:
1. -8 and 10 (Southbridge)
2. $(-2)^n$ (St. John's) $(-2)^n$
3. 2 (Quaboag)

TEAM Round
1. 3 (St. John's)
2. 2.5 or $2\frac{1}{2}$ or $\frac{5}{2}$ (Mass Academy)
3. 351 (Bancroft)
4. 58 (Auburn)
5. $-\frac{1}{17}$ or ~~-0.0588~~ (Norton)
6. $\frac{5}{7}$ or 0.714 (Bartlett)
7. 5 (Auburn)
8. $s+r$ (Shepherd Hill)
9. 8 (Athol)

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 4, 2017

Round 1: Arithmetic

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Let $a \# b = a^2b - ab^2$. If $a = 3$ and $b = -2$, evaluate: $\frac{(a \# b)^2}{15-(a \# b)}$.

2. Simplify: $\frac{18-\left[22-3\left(3^2-\sqrt{289}\right)\right]}{2^2+45\div9-2}$

3. Evaluate: $504^3 - 503^3$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 4, 2017

Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. An 80 foot rope is cut into 2 pieces. One piece is 23 feet shorter than the other. What is the length of the larger piece of rope?

2. A pool is $\frac{3}{4}$ full of water. 120 gallons of water is added so that the pool is now $\frac{15}{16}$ full. How many gallons of water are in the pool when it is $\frac{1}{2}$ full?

3. Greg takes half as long as Sherman to do a job, and Greg takes three times as long as Rob to do the same job. If all three men can do the job together in 12 days, how long would it take Rob to do the job alone?

ANSWERS

(1 pt.) 1. ______________________ feet

(2 pts.) 2. ______________________ gallons

(3 pts.) 3. ______________________ days

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 4, 2017
Round 3: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

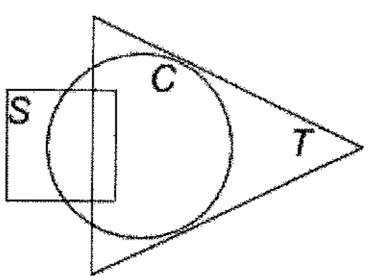
1. Let $M = \{1, 2, 3, ..., 10\}$, $D = \{1, 3, 5, 7, ...\}$ and $P = \{2, 4, 6, 8, 10\}$.
   Find $(M \cap P) \cup (D \cap P)$.

2. If S' denotes the complement of the set S, shade the set $(S' \cap T) \cup (C \cap S' \cap T')$ on the diagram given on the answer line below.

3. How many subsets of the set $\{a, b, c, d, e, f, g\}$ have 3 or fewer elements?

ANSWERS

(1 pt.) 1. ____________________



(2 pts.) 2.

(3 pts.) 3. ____________________ subsets

WORCESTER COUNTY MATHEMATICS LEAGUE

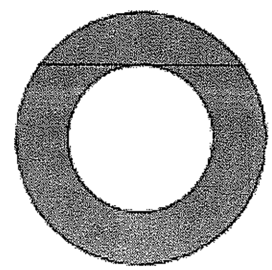
Varsity Meet 1 - October 4, 2017
Round 4: Measurement

All answers must be in simplest exact form in the answer section

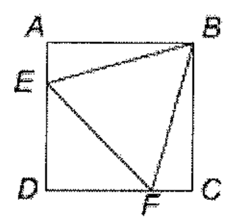
NO CALCULATOR ALLOWED

1. A cylinder has a radius of $4\ m$ and a volume of $144\pi\ m^3$. What is the altitude of the cylinder?

2. If the chord shown is 18 cm long and is tangent to the smaller circle, what is the area of the shaded region?



3. In this figure, $ABCD$ is a square and EBF is an equilateral triangle. If the area of $ABCD$ is 1 square unit, what is the area of EBF ?



ANSWERS

(1 pt.) 1. ______________________ m

(2 pts.) 2. ______________________ square cm

(3 pts.) 3. ______________________ square units

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 4, 2017

Round 5: Polynomial Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve $r^2 - 2r - 80 = 0$.

2. If the product of the zeroes of the n^{th} degree polynomial $P(x)$ is 1 and $P(x)$ has n distinct real roots, determine the product of the roots of $P(-0.5x)$

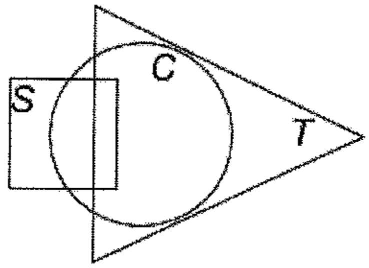
3. What rational number m will allow $x - m$ to divide into $x^3 - 2x^2 + 2x - 10$ such that the remainder from the division will be equal to $-3m$?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________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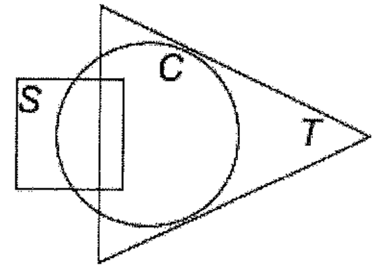
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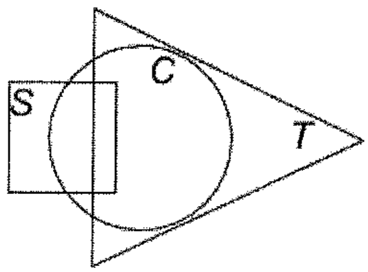
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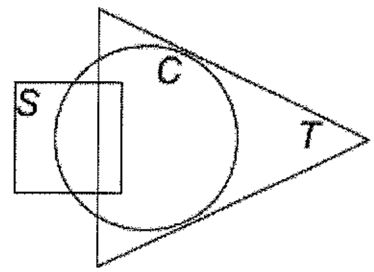
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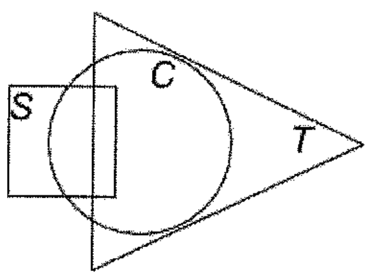
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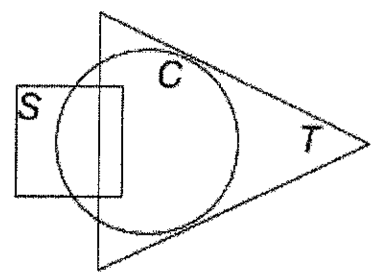
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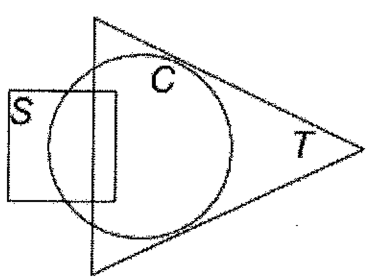
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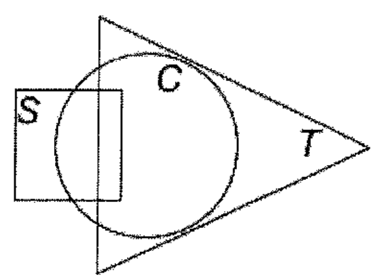

S
C
T

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 4, 2017

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 points each)

APPROVED CALCULATORS ALLOWED

1. If the six digit number $2a25b7$ is divisible by 99, what is the value of $a-b$?

2. If $x+y=1$ and $x^2+y^2=2$, find the numerical value of x^3+y^3.

3. A proper subset of a set X is any subset of X that is not equal to X itself. How many proper subsets of $\{a,b,c,d,e,\ldots,z\}$ have 24 or more elements?

4. There is a tree that is 6 feet in circumference and 40 feet tall. A vine winds seven times around the trunk of the tree and reaches the top of the tree. What is the minimum length of the vine?

5. A cubic polynomial $f(x)$ has real coefficients and has zeroes at 2 and $2+i$. If $f(-2)=4$, then determine the coefficient of x^3 in the standard form of $f(x)$.

6. In a certain population of swans, $\frac{1}{3}$ of the males have mates and $\frac{1}{4}$ of the females have mates. Assuming that all mating partners come from this population, what fraction of the population does not have a mate?

7. Solve for x: $\frac{x+6}{x-3}+\frac{x}{5}=\frac{x(x+3)}{5(x-3)}-\frac{5}{3-x}$

8. Simplify: $\dfrac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{rs}}$

9. The sum of the squares of 6 positive consecutive integers is 199. What is the largest of these six numbers?

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 4, 2017 - SOLUTIONS

Round 1: Arithmetic

1. Let $a \# b = a^2b - ab^2$. If $a = 3$ and $b = -2$, evaluate: $\frac{(a \# b)^2}{15-(a \# b)}$.

Solution: We have that

$$\frac{(a \# b)^2}{15-(a \# b)} = \frac{(3 \# (-2))^2}{15-(3 \# (-2))}.$$

To begin, let's evaluate the term

$$(3 \# (-2)) =$$
$$3^2(-2) - 3(-2)^2 =$$
$$-18 - 12 = -30.$$

Now we can plug in to get that

$$\frac{(3 \# (-2))^2}{15-(3 \# (-2))} =$$
$$\frac{(-30)^2}{15-(-30)} =$$
$$\frac{900}{45} = 20.$$

2. Simplify: $\frac{18-\left[22-3\left(3^2-\sqrt{289}\right)\right]}{2^2+45\div9-2}$

Solution: We have that

$$\frac{18-\left[22-3\left(3^2-\sqrt{289}\right)\right]}{2^2+45\div9-2} =$$
$$\frac{18-[22-3(9-17)]}{2^2+5-2} =$$
$$\frac{18-[22-3(-8)]}{4+3} =$$
$$\frac{18-[22+24]}{7} =$$
$$\frac{-28}{7} = -4.$$

3. Evaluate: $504^3 - 503^3$

Solution 1: Using the formula for difference of cubes, we have that

$$\begin{aligned}
&504^3 - 503^3 = \\
&(504 - 503)\left(504^2 + (504)(503) + 503^2\right) = \\
&(1)\left(504^2 + (504)(503) + 503^2\right) = \\
&504^2 + (504)(503 + 1) - 504 + 503^2 = \\
&2\left(504^2\right) + 503^2 - 504. \qquad (1)
\end{aligned}$$

Now we note that for an integer c we have that

$$(500 + c)^2 = 500^2 + 1000c + c^2 = 250000 + 1000c + c^2.$$

This gives us that

$$504^2 = (500 + 4)^2 = 250000 + 4000 + 16 = 254016 \qquad (2)$$

$$503^2 = (500 + 3)^2 = 250000 + 3000 + 9 = 253009 \qquad (3)$$

Plugging equations (2) and (3) into equation (1) gives us that

$$504^3 - 503^3 = 2(254016) + 253009 - 504 = 760537.$$

Solution 2 (Multiply and Subtract):

$$\begin{aligned}
504^3 &= \quad 128{,}024{,}064 \\
-\ 503^3 &= -\ 127{,}253{,}527 \\
\hline
&\qquad 760{,}537
\end{aligned}$$

Solution 3: We know that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. We can make use of this fact by slightly rearranging the given expression:

$$\begin{aligned}
&504^3 - 503^3 = \\
&(503 + 1)^3 - 503^3 = \\
&[503^3 + (3 \times 503^2) + (3 \times 503) + 1] - 503^3 = \\
&3 \times 503^2 + 3 \times 503 + 1 = \\
&759{,}027 + 1{,}509 + 1 = 760{,}537
\end{aligned}$$

Round 2: Algebra 1

1. An 80 foot rope is cut into 2 pieces. One piece is 23 feet shorter than the other. What is the length of the larger piece of rope?

Solution: Let x be the length of the larger piece of rope. We have that

$$x+(x-23)=80$$
$$2x=103$$
$$x=\tfrac{103}{2}.$$

2. A pool is $\frac{3}{4}$ full of water. 120 gallons of water is added so that the pool is now $\frac{15}{16}$ full. How many gallons of water are in the pool when it is $\frac{1}{2}$ full?

Solution 1: Let x be the number of gallons in the pool when it is full. We have that

$$\tfrac{3}{4}x+120=\tfrac{15}{16}x$$
$$12x+120\times 16=15x$$
$$3x=1920$$
$$x=640.$$

Since the question asks how many gallons are in the pool when it is half full, we have that the desired answer is $\frac{1}{2}x=320$.

Solution 2: Since $\frac{3}{4}$ is equivalent to $\frac{12}{16}$ we know that 120 gallons must be precisely $\frac{3}{16}$ of the total volume of the pool. Therefore, 40 gallons fills $\frac{1}{16}$ of the pool. Hence, when the pool is half full there must be $40\times 8=320$ gallons of water.

3. Greg takes half as long as Sherman to do a job, and Greg takes three times as long as Rob to do the same job. If all three men can do the job together in 12 days, how long would it take Rob to do the job alone?

Solution 1: Let g denote the rate of work that Greg performs, with units jobs/day. Similarly let s and r denote the rates of work for Sherman and Rob. We are given that

$$g=2s \qquad (1)$$
$$r=3g \qquad (2)$$
$$(g+s+r)\times(12\ days)=1\ job. \qquad (3)$$

Plugging in equations (1) and (2) into equation (3) gives us that

$$(g + \tfrac{1}{2}g + r) \times (12\ days) = 1\ job.$$
$$(\tfrac{1}{3}r + \tfrac{1}{6}r + r) \times (12\ days) = 1\ job.$$
$$(\tfrac{3}{6}r + r) \times (12\ days) = 1\ job.$$
$$(\tfrac{3}{2}r) \times (12\ days) = 1\ job.$$
$$r \times 18\ days\ = 1\ job$$
$$r\ =\ \tfrac{1\ job}{18\ days}.$$

Therefore, our desired answer is 18 days.

Solution 2: Let x be the number of days that it takes Rob to get the job done. Therefore, we know that in 1 day Rob completes $\frac{1}{x}$ of the job.

Guy gets the job done in $3x$ days so in 1 day completes $\frac{1}{3x}$ of the job.

Sherman gets the job done in $6x$ days so in 1 day he completes $\frac{1}{6x}$ of the job.

In one day the three men working together complete

$$\tfrac{1}{x} + \tfrac{1}{3x} + \tfrac{1}{6x} =$$
$$\tfrac{6}{6x} + \tfrac{2}{6x} + \tfrac{1}{6x} =$$
$$\tfrac{9}{6x} = \tfrac{3}{2x} \text{ of the job.}$$

The job takes 12 days to complete, meaning that

$$12 \times \tfrac{3}{2x} = 1$$
$$x = 6 \times 3 = 18 \text{ days.}$$

Round 3: Set Theory

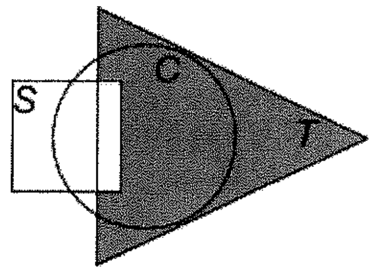
1. Let $M = \{1,\ 2,\ 3,\ \ldots,\ 10\}$, $D = \{1,\ 3,\ 5,\ 7,\ \ldots\}$ and $P = \{2,\ 4,\ 6,\ 8,\ 10\}$. Find $(M \cap P) \cup (D \cap P)$.

   Solution: Work inside the parentheses first. We see that

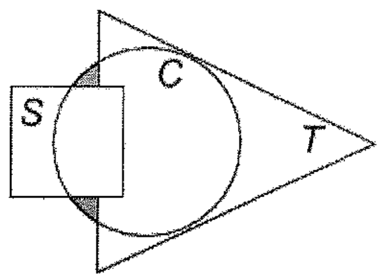
$$(M \cap P) \cup (D \cap P) =$$
$$\{2,\ 4,\ 6,\ 8,\ 10\} \cup \{\varnothing\} =$$
$$\{2,\ 4,\ 6,\ 8,\ 10\} = P.$$

2. If S' denotes the complement of the set S, shade the set $(S' \cap T) \cup (C \cap S' \cap T')$ on the diagram given on the answer line below.

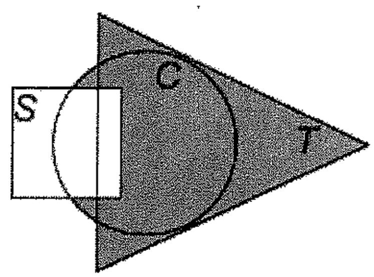
Solution 1: Work inside the parentheses first. We have that $(S' \cap T)$ is the set of all points that are contained in T and *not* contained in S. We can shade this area as

$(S' \cap T) =$ 

Next we have that $(C \cap S' \cap T')$ is the set of all points in C that are in neither S nor T. We can shade this area as

$(C \cap S' \cap T') =$ 

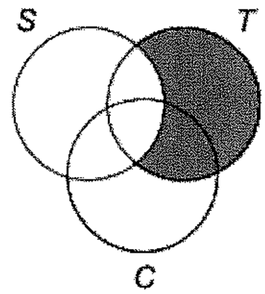
The union of these two above sets can be shaded as

$(S' \cap T) \cup (C \cap S' \cap T') =$ 

Solution 2: Notice that the given diagram is essentially equivalent to a traditional 3-circle Venn diagram. We know this since each region has its own unique area, an area it shares with one of each of the other regions, and an area that it shares with both other regions. Therefore, we can solve the problem as-if it used a Venn diagram:

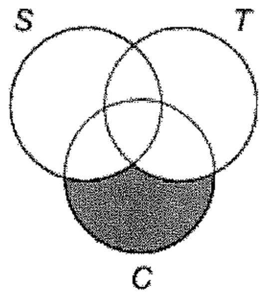
Work inside the parentheses first. We have that $(S' \cap T)$ is the set of all points that are contained in T and *not* contained in S. We can shade this area as

$(S' \cap T) =$



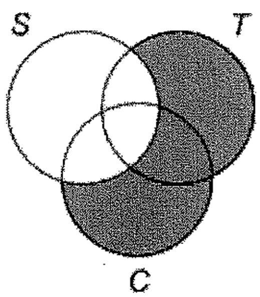
Next we have that $(C \cap S' \cap T')$ is the set of all points in C that are in neither S nor T. We can shade this area as

$(C \cap S' \cap T') =$



The union of these two above sets can be shaded as

$(S' \cap T) \cup (C \cap S' \cap T') =$



Hence, we simply need to shade $S' \cap (T \cup C)$ in the provided diagram.

3. How many subsets of the set $\{a, b, c, d, e, f, g\}$ have 3 or fewer elements?

Solution 1: To determine the answer to this question, we must determine the sum of the number of subsets with 0 elements, the number of subsets with 1 element, the number of subsets with 2 elements and the number of subsets with 3 elements.

The only subset with zero elements is the empty set.

The subsets with one element are simply $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$, $\{f\}$, $\{g\}$.

To determine the number of subsets with two elements, we need to determine how many ways we can pick two objects from a collection of seven. This number is expressed by

$$\binom{7}{2} = \frac{7!}{2! \times 5!} = \frac{7\times 6}{2} = 21.$$

To determine the number of subsets with three elements, we need to determine how many ways we can pick three objects from a collection of seven. This number is expressed by

$$\binom{7}{3} = \frac{7!}{3! \times 4!} = \frac{7\times 6\times 5}{3\times 2\times 1} = 35.$$

Therefore, the desired answer is $35 + 21 + 7 + 1 = 64$.

Solution 2: The number of subsets with 3 or fewer elements is given by:

(# of ways to pick 0 objects from 7) + (# of ways to pick 1 object from 7) +
+ (# of ways to pick 2 objects from 7) + (# of ways to pick 3 objects from 7) =

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3}$$

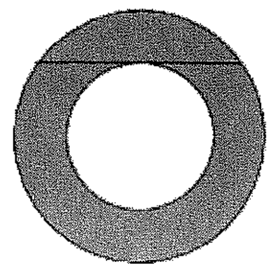
$$= \quad 1 \quad + \quad 7 \quad + \quad 21 \quad + \quad 35 = 64$$

Round 4: Measurement

1. A cylinder has a radius of $4\ m$ and a volume of $144\pi\ m^3$. What is the altitude of the cylinder?

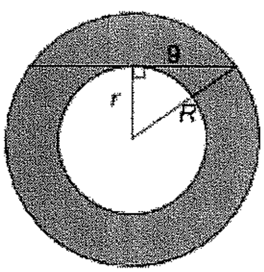
   Solution: We know that the volume of a cylinder is given by the equation
   $$V = \pi r^2 \times a.$$

   Therefore, we have that
   $$144\pi = \pi(4)^2 \times a$$
   $$a = \tfrac{144}{16} = 9.$$



2. If the chord shown is 18 cm long and is tangent to the smaller circle, what is the area of the shaded region?

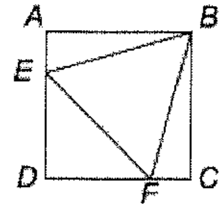
   Solution: We can draw in some information on the diagram. If we let r denote the radius of the small circle and R denote the radius of the large circle, we have that:



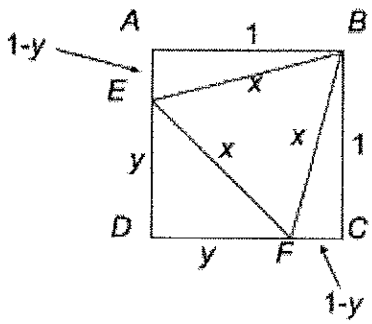
Now the desired area is given by $\pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right)$.

Notice that from the diagram we can deduce using the Pythagorean Theorem that $r^2 + 9^2 = R^2$, or rewritten as $R^2 - r^2 = 81$. Therefore we have that the desired area is given by $\pi\left(R^2 - r^2\right) = 81\pi$.

3. In this figure, $ABCD$ is a square and EBF is an equilateral triangle. If the area of $ABCD$ is 1 square unit, what is the area of EBF ?



Solution: Let x be the side length of the equilateral triangle and let y be the length of segment DF. Drawing in the information we have gives us:



From this diagram we can easily determine the area of the region within the square that is not in the triangle. This area is $\frac{1}{2}y^2 + 2\left(\frac{1-y}{2}\right) = 1 - y + \frac{1}{2}y^2$. Since the area of the square is 1, we have that the area of the triangle must be

$$1 - \left(1 - y + \tfrac{1}{2}y^2\right) = y - \tfrac{1}{2}y^2 = y\left(1 - \tfrac{1}{2}y\right).$$

Notice now that we can now deduce two equations relating x and y. We have that

$$y^2 + y^2 = x^2 \qquad (1)$$
$$(1-y)^2 + 1^2 = x^2 \qquad (2)$$

Combining equations (1) and (2) we see that

$$2y^2 = 1 - 2y + y^2 + 1$$
$$y^2 + 2y - 2 = 0$$

Using the quadratic equation, we have that

$$y = \frac{-2 \pm \sqrt{4+8}}{2} =$$
$$y = -1 \pm \sqrt{3}$$

Since y must be positive, we have that $y = \sqrt{3} - 1$. Recall that we determined that the area of the triangle must be $y\left(1 - \frac{1}{2}y\right)$. Therefore, the triangle's area is

$$\left(\sqrt{3} - 1\right)\left(1 - \tfrac{1}{2}\left(\sqrt{3} - 1\right)\right) =$$
$$\left(\sqrt{3} - 1\right)\left(\tfrac{3}{2} - \tfrac{1}{2}\sqrt{3}\right) =$$
$$\tfrac{3}{2}\sqrt{3} - \tfrac{3}{2} - \tfrac{3}{2} + \tfrac{1}{2}\sqrt{3} =$$
$$2\sqrt{3} - 3$$

Round 5: Polynomial Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve $r^2 - 2r - 80 = 0$.

Solution 1 (Factor): We have that

$$r^2 - 2r - 80 = 0$$
$$(r - 10)(r + 8) = 0$$
$$r = 10 \text{ or } r = -8.$$

Solution 2 (Quadratic Equation): Using the quadratic equation we have that

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-80)}}{2(1)}$$
$$r = \frac{2 \pm \sqrt{4+320}}{2}$$
$$r = \frac{2 \pm 2\sqrt{81}}{2} = 1 \pm 9 = 10 \text{ or } -8$$

2. If the product of the zeroes of the n^{th} degree polynomial $P(x)$ is 1 and $P(x)$ has n distinct real roots, determine the product of the roots of $P(-0.5x)$

Solution: Let $r_1, r_2, \ldots, r_n$ be the zeroes of $P(x)$. We have for any of these zeroes, r_i, that $P(r_i) = 0$. It is also the case then that $P((-0.5)(-2)r_i) = P(r_i) = 0$. This means that for each r_i, there is a corresponding zero of $P(-0.5x)$ of $-2r_i$.

Since we have that $r_1 \times r_2 \times \cdots \times r_n = 1$, we have that $(-2)r_1 \times (-2)r_2 \times \cdots \times (-2)r_n = (-2)^n$. Hence, the desired product is $(-2)^n$.

3. What rational number m will allow $x-m$ to divide into $x^3-2x^2+2x-10$ such that the remainder from the division will be equal to $-3m$?

Solution: Using synthetic division, we have that

$$\begin{array}{c|cccc} m & 1 & -2 & 2 & -10 \\ & & m & m^2-2m & m^3-2m^2+2m \\ \hline & 1 & m-2 & m^2-2m+2 & m^3-2m^2+2m-10 \end{array}$$

Now we simply need to solve

$$-3m = m^3-2m^2+2m-10$$
$$m^3-2m^2+5m-10=0$$
$$m^2(m-2)+5(m-2)=0$$
$$(m^2+5)(m-2)=0$$

The only rational solution to this equation is $m=2$.

Team Round

1. If the six digit number $2a25b7$ is divisible by 99, what is the value of $a-b$?

Solution 1: If the number is divisible by 99 then it is also divisible by 9. For the number to be divisible by 9 the sum of its digits must be divisible by 9. Hence, $2+a+2+5+b+7=16+a+b$ must be divisible by 9. Therefore, $a+b$ must either be equal to 2 or equal to 11.

We also know that if the number is divisible by 99 it is also divisible by 11. If a number is divisible by 11, then the alternating sum of its digits must also be divisible by 11. That is:

$$2-a+2-5+b-7= \ -8-(a-b)= \ -1(8+(a-b))$$

We therefore have that $8+(a-b)$ must be divisible by 11. The only ways we can make this expression equal to 0 is if $a=0, b=8$ or if $a=1, b=9$. But neither of these possibilities satisfy the requirement that $a+b$ must either be equal to 2 or equal to 11.

Hence, we have that $a-b=3$, which is the desired result.

Solution 2: Using the rule for divisibility by 11 seen in solution 1, we have that for some integer n:

$$\begin{aligned}2+2+b &= a+5+7\pm 11n\\ 4+b &= a+12\pm 11n\\ b-a &= 8\pm 11n\end{aligned}$$

Since a and b are single digits, their difference cannot be greater than 10. Hence, n must be equal to -1, giving us that

$$b-a=\ -3\ \rightarrow\ a-b=3.$$

2. If $x+y=1$ and $x^2+y^2=2$, find the numerical value of x^3+y^3.

Solution: If $x+y=1$ then we have that

$$\begin{aligned}(x+y)^2 &= 1^2\\ x^2+2xy+y^2 &= 1.\end{aligned}$$

Since we are given that $x^2+y^2=2$, we therefore know that

$$\begin{aligned}2+2xy &= 1\\ 2xy &= \ -1\\ xy &= \ -\tfrac{1}{2}.\end{aligned}$$

Now we note that $x^3+y^3=(x+y)(x^2-xy+y^2)$. Plugging in our known values we have that

$$\begin{aligned}x^3+y^3 &= (x+y)(x^2-xy+y^2)\\ &= (1)(2+\tfrac{1}{2})=\tfrac{5}{2}.\end{aligned}$$

3. A proper subset of a set X is any subset of X that is not equal to X itself. How many proper subsets of $\{a,b,c,d,e,\ldots,z\}$ have 24 or more elements?

Solution: Since there are 26 elements in this set, we know that no proper subset will have 26 or more elements. Therefore, we need to compute the number of subsets which have either 25 or 24 elements in them.

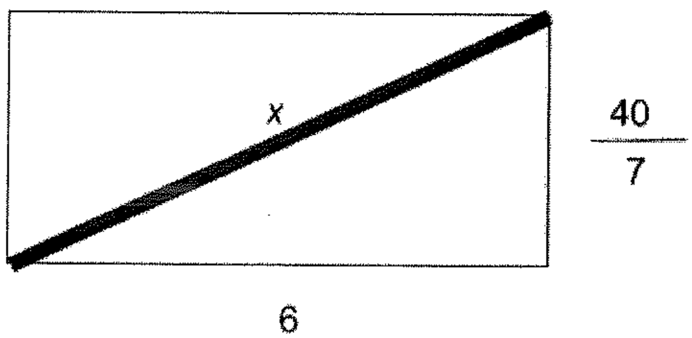
This is given precisely by: $\binom{26}{25}+\binom{26}{24}$

Using the n-choose-k formula we have that the above expression evaluates to

$$\frac{26!}{(25!)(1!)}+\frac{26!}{(24!)(2!)}=26+\frac{26\times 25}{2}=351.$$

4. There is a tree that is 6 feet in circumference and 40 feet tall. A vine winds seven times around the trunk of the tree and reaches the top of the tree. What is the minimum length of the vine?

Solution 1: As it does not affect the calculation assume that the vine climbs the tree in a linear fashion. That is, we assume that once the vine has wrapped around the tree one time it will be precisely $\frac{40}{7}$ feet high. This can be represented graphically by:

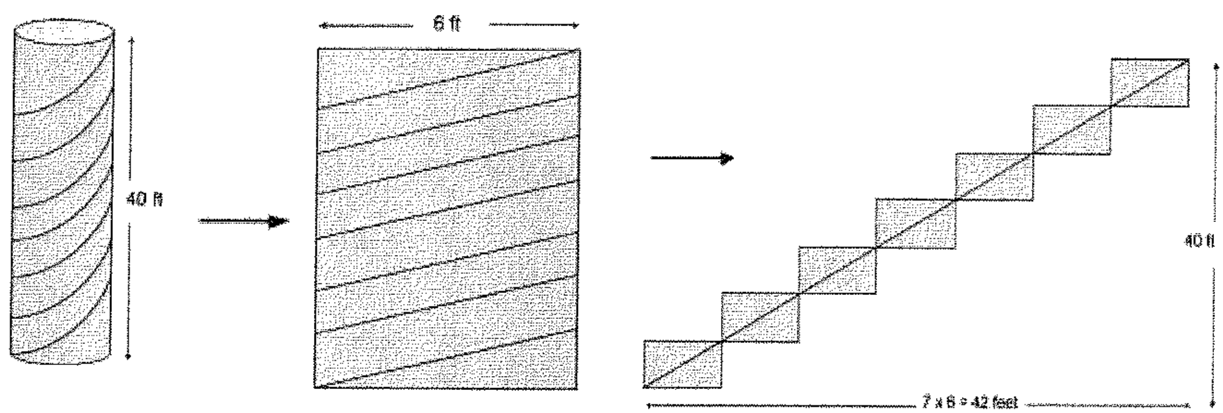


Since the vine must wrap around the tree seven times, the minimum length of the vine is $7x$. If we multiply each side in the above diagram by 7, the Pythagorean theorem gives us that

$$42^2 + 40^2 = (7x)^2$$
$$3364 = (7x)^2$$
$$7x = 58.$$

Hence, the minimum length of the vine is 58 feet.

Solution 2: Begin by drawing a rough picture of the tree as a cylinder. Then consider unrolling the cylinder into a rectangle. Since the tree has a circumference of 6 feet, this rectangle is 40 ft tall and 6ft wide. Now imagine slicing this rectangle horizontally into seven even strips and then line them up corner to corner:



From the diagram we can see that the vine is simply the hypotenuse of a right triangle with legs 40 ft and 42ft. The Pythagorean Theorem gives us that the vine is 58 ft long.

5. A cubic polynomial $f(x)$ has real coefficients and has zeroes at 2 and $2+i$. If $f(-2)=4$, then determine the coefficient of x^3 in the standard form of $f(x)$.

Solution: Since f has a zero of $2+i$ we know it must also have a zero of $2-i$. This means we can write for some constant c that

$$f(x) = c(x-2)(x-(2+i))(x-(2-i))$$
$$f(x) = c(x-2)(x^2-4x+5)$$
$$f(x) = c(x^3-6x^2+13x-10)$$

Since $f(-2)=4$, we have that

$$4 = c[(-2)^3 - 6(-2)^2 + 13(-2) - 10]$$
$$4 = -68c$$
$$c = -\tfrac{4}{68} = -\tfrac{1}{17}.$$

Since the coefficient of x^3 in $f(x)$ is simply c, we have that the desired answer is $-\frac{1}{17}$.

6. In a certain population of swans, $\frac{1}{3}$ of the males have mates and $\frac{1}{4}$ of the females have mates. Assuming that all mating partners come from this population, what fraction of the population does not have a mate?

Solution 1: Let x be the number of male swans and y be the number of female swans in the population. From the given information, we have that $\frac{1}{3}x = \frac{1}{4}y$. This means that $x = \frac{3}{4}y$. The number of swans without a mate is

$$\tfrac{2}{3}x + \tfrac{3}{4}y =$$
$$\tfrac{2}{3}\left(\tfrac{3}{4}y\right) + \tfrac{3}{4}y =$$
$$\tfrac{1}{2}y + \tfrac{3}{4}y = \tfrac{5}{4}y.$$

The total number of swans is $x+y = \frac{3}{4}y + y = \frac{7}{4}y$. Therefore, the fraction of swans without a mate is

$$\frac{\frac{5}{4}y}{\frac{7}{4}y} = \frac{5}{7}.$$

Solution 2: Suppose that there are 12 male swans. Since one third of the males have a mate, that means that 4 male swans have a mate. Hence, 4 females have a mate. If those 4 females represent one quarter of the female swans, we know there are 16 females.

Hence, there are a total of 28 swans and 20 do not have mates. The desired fraction therefore is

$$\frac{20}{28} = \frac{5}{7}.$$

7. Solve for x : $\frac{x+6}{x-3} + \frac{x}{5} = \frac{x(x+3)}{5(x-3)} - \frac{5}{3-x}$

Solution: Multiply both sides of the equation by $5(x-3)$ to get

$$\begin{aligned}
&\frac{x+6}{x-3} + \frac{x}{5} = \frac{x(x+3)}{5(x-3)} - \frac{5}{3-x} \\
&5(x+6) + (x-3)x = x(x+3) + 5^2 \\
&5x + 30 + x^2 - 3x = x^2 + 3x + 25 \\
&2x + 30 = 3x + 25 \\
&x = 5.
\end{aligned}$$

8. Simplify: $\frac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{rs}}$

Solution: We have that

$$\begin{aligned}
&\frac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{rs}} = \\
&\frac{\frac{s}{rs}+\frac{r}{rs}}{\frac{1}{rs}} = \\
&\frac{\frac{s+r}{rs}}{\frac{1}{rs}} = \frac{s+r}{rs} \times \frac{rs}{1} = s + r.
\end{aligned}$$

9. The sum of the squares of 6 positive consecutive integers is 199. What is the largest of these six numbers?

Solution: Let x be the smallest of the six consecutive numbers. The described sum can be written $x^2 + (x+1)^2 + \ldots + (x+5)^2$. Expanding each term we have

$$x^2 + (x^2 + 2x + 1) + (x^2 + 4x + 4) + (x^2 + 6x + 9) + (x^2 + 8x + 16) + (x^2 + 10x + 25) =$$
$$6x^2 + 30x + 55 = 199$$
$$6x^2 + 30x - 144 = 0$$
$$6(x^2 + 5x - 24) = 0$$
$$(x+8)(x-3) = 0.$$

Since the numbers must be positive, we have that $x = 3$. Therefore, the largest of the six numbers is 8.